ELSEVIER

Discrete Applied Mathematics 79 (1997) 35–44

DISCRETE
APPLIED
MATHEMATICS

The complexity of recognizing tough cubic graphs

D. Bauer[a,*,1], J. van den Heuvel[b,2,3], A. Morgana[c], E. Schmeichel[d,4]

[a] Department of Mathematical Sciences, Stevens Institute of Technology, Hoboken, NJ 07030, USA
[b] Department of Mathematics and Statistics, Simon Fraser University, Burnaby, BC, Canada V5A 1S6
[c] University of Rome, Rome, Italy
[d] Department of Mathematics & Computer Science, San Jose State University, San Jose, CA 95192, USA

Received 31 August 1995; received in revised form 15 August 1996; accepted 5 March 1997

Abstract

We show that it is NP-hard to determine if a cubic graph G is 1-tough. We then use this result to show that for any integer $t \geqslant 1$, it is NP-hard to determine if a $3t$-regular graph is t-tough. We conclude with some remarks concerning the complexity of recognizing certain subclasses of tough graphs.

AMS classification: 68R10; 05C38

Keywords: Toughness; Cubic graphs; NP-completeness

1. Introduction

We begin with a few definitions and some notation. A good reference for any undefined terms is [7]. We consider only undirected graphs with no loops or multiple edges. Let $\omega(G)$ denote the number of components of a graph G. A graph G is *t-tough* if $|S| \geqslant t \cdot \omega(G - S)$ for every subset S of the vertex set $V(G)$ of G with $\omega(G - S) > 1$. The *toughness* of G, denoted $\tau(G)$, is the maximum value of t for which G is t-tough (taking $\tau(K_n) = \infty$ for all $n \geqslant 1$). A *k-factor* is a k-regular spanning subgraph. Of course, a hamiltonian cycle is a connected 2-factor. We use $\delta(G)$ to denote the minimum vertex degree in G.

* Corresponding author. E-mail: bauer@hamilton.stevens-tech.edu.

[1] Supported partially by NATO Collaborative Research Grant CRG 921251.

[2] Supported by a Grant from the Natural Sciences and Engineering Council of Canada.

[3] Current address: Centre for Discrete and Applicable Mathematics, Department of Mathematics, London School of Economics, Houghton Street, London WC2A 2AE, England, UK.

[4] Supported in part by the National Science Foundation under Grant DMS9206991.

PII S0166-218X(97)00030-9

Toughness was introduced by Chvátal in [9]. An obvious connection between toughness and hamiltonicity is that being 1-tough is a necessary condition for a graph to be hamiltonian. Chvátal raised a number of interesting questions concerning toughness. He conjectured that every k-tough graph on n vertices with $n \geqslant k+1$ and kn even has a k-factor. It has been established that this is both true and best possible.

Theorem 1.1 (Enomoto et al. [15]). *Let $k \geqslant 1$.*

(i) *If G is a k-tough graph on n vertices with $n \geqslant k+1$ and kn even, then G has a k-factor.*

(ii) *For any $\varepsilon > 0$, there exists an infinite family of $(k-\varepsilon)$-tough graphs with kn even and no k-factor.*

Chvátal also conjectured that there exists a constant t_0 such that every t_0-tough graph is hamiltonian. This is still open. The smallest t_0 for which this might be true is $t_0 = 2$. This follows by letting $k = 2$ in Theorem 1.1 (ii) above.

The problem of determining the complexity of recognizing t-tough graphs was first raised by Chvátal [8] and later appeared in [23, 10, p. 429]. Consider the following decision problem, where $t \geqslant 1$ is a rational number.

t-TOUGH

Instance: Graph G.

Question: Is $\tau(G) \geqslant t$?

The following was established in [3].

Theorem 1.2 (Bauer et al. [3]). *For any rational $t \geqslant 1$, t-TOUGH is* NP*-hard.*

It seems natural to inquire whether the problem of recognizing t-tough graphs remains NP-hard for various subclasses of graphs. For example, the following theorem of Dirac [14] might suggest that the problem of recognizing 1-tough graphs will be easier for dense graphs; i.e., graphs with many edges.

Theorem 1.3 (Dirac [14]). *Let G be a graph on $n \geqslant 3$ vertices with $\delta(G) \geqslant \frac{1}{2}n$. Then G is hamiltonian.*

In [4], a relatively satisfying solution for the complexity of recognizing t-toughness in dense graphs was obtained. Let $\Omega(r)$ denote the class of all graphs G satisfying $\delta(G) \geqslant r\,|V(G)|$.

Theorem 1.4 (Bauer et al. [4]). *Let $t \geqslant 1$ be any rational number.*

(i) *Every graph in $\Omega((t)/(t+1))$ is t-tough.*

(ii) *For any $\varepsilon > 0$, t-TOUGH remains* NP*-hard for graphs in $\Omega(t/(t+1)-\varepsilon)$.*

Of course, Theorem 1.4 (i) for $t = 1$ is an immediate consequence of Theorem 1.3.

Another interesting class of graphs is the class of bipartite graphs. Obviously $\tau(G) \leqslant 1$ for any bipartite graph G. The complexity of recognizing 1-tough, bipartite graphs has been raised by several authors; see, e.g., [6, p. 119], and we have the following recent result.

Theorem 1.5 (Kratsch et al. [19]). *1-TOUGH remains* NP-*hard for bipartite graphs.*

On the other hand, there exist classes of graphs for which it is NP-hard to determine whether a graph in the class is hamiltonian, but polynomial to determine if it is 1-tough. One such class is the class of split graphs. A graph G is called a *split graph* if $V(G)$ can be partitioned into an independent set and a clique. Determining if a split graph is hamiltonian was shown to be NP-hard in [11]. On the other hand, the following was shown in [19].

Theorem 1.6 (Kratsch et al. [19]). *The class of* 1*-tough split graphs* $G(V,E)$ *can be recognized in* $O(|E|\sqrt{|V|})$ *time.*

Our main interest in this paper is the complexity of toughness for the class of cubic graphs. Chvátal [9] showed that a necessary and sufficient condition on the integer n for the existence of an n-vertex, $\frac{3}{2}$-tough cubic graph is that $n=4$ or $n \equiv 0 \pmod 6$. Later, Jackson and Katerinis [18] strengthened this by characterizing $\frac{3}{2}$-tough, cubic graphs.

Theorem 1.7 (Jackson and Katerinis [18]). *Let G be a cubic graph. Then G is $\frac{3}{2}$-tough if and only if $G=K_4$, $G=K_2 \times K_3$, or G is obtained from a* 3*-connected cubic graph by replacing all the vertices of this graph by triangles.*

The above characterization allows $\frac{3}{2}$-tough, cubic graphs to be recognized in polynomial time. However, the arguments used in [18] seem very dependent on the particular constant $\frac{3}{2}$, and it seems unlikely that a polynomial algorithm exists to recognize t-tough, cubic graphs for any t such that $1 \leqslant t < \frac{3}{2}$. In fact, the following theorem in support of this assertion is our main result.

Theorem 1.8. *1-TOUGH remains* NP-*hard for cubic graphs.*

Using Theorem 1.8 we also prove the following more general result.

Theorem 1.9. *For any integer $t \geqslant 1$, t-TOUGH remains* NP-*hard for $3t$-regular graphs.*

The following conjecture of Goddard and Swart [17] is related to Theorem 1.9.

Conjecture 1.10 (Goddard and Swart [17]). *Let G be a k-regular graph. Then G is $\frac{1}{2}k$-tough if and only if G is k-connected and $K_{1,3}$-free.*

The "if" direction of Conjecture 1.10 was established by the following result of Matthews and Sumner [20].

Theorem 1.11 (Matthews and Sumner [20]). *Let G be a noncomplete $K_{1,3}$-free graph. Then $\tau(G)$ is equal to one half of the connectivity of G.*

More evidence for the truth of Conjecture 1.10 was given in Theorem 1.7; in particular, Theorem 1.7 clearly implies the truth of Conjecture 1.10 for $k=3$. If Conjecture 1.10 is true, then t-tough, $2t$-regular graphs can be recognized in polynomial time. By contrast, we now make the following conjecture.

Conjecture 1.12. *For any integer $t \geqslant 1$, t-TOUGH remains* NP-*hard for* $(2t+1)$-*regular graphs.*

We present the proofs of Theorems 1.8 and 1.9 and a possible approach to Conjecture 1.12 in the following section. We then conclude the paper by discussing several open problems on the complexity of recognizing certain subclasses of tough graphs.

2. Proofs of Theorems 1.8 and 1.9

Proof of Theorem 1.8. We will prove that 1-TOUGH is NP-hard for cubic graphs by reducing 1-TOUGH for general connected graphs with at least two vertices.

Let $G \neq K_1$ be any connected graph. Construct the corresponding cubic graph $H = H(G)$ as follows. Each vertex $v \in V(G)$ will correspond to the graph H_v in Fig. 1 below, in which there are $d_G(v)$ black vertices of degree 2 on each side of the indicated edge e. Arbitrarily designate the black vertices in H_v on one side of e by A_v, and those on the other side by B_v. Note that every vertex in $H_v - (A_v \cup B_v)$ belongs to a triangle in H_v.

An edge vw in G will be represented in H by joining any previously unused vertex in A_v (i.e., one whose degree is then 2) to any previously unused vertex in B_w, and any previously unused vertex in A_w to any previously unused vertex in B_v.

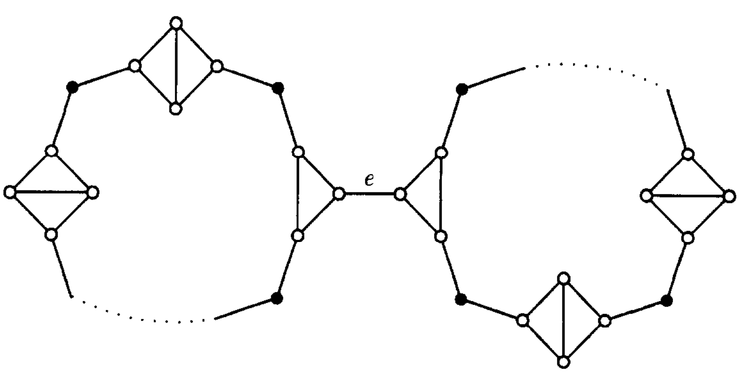

Fig. 1. The graph H_v.

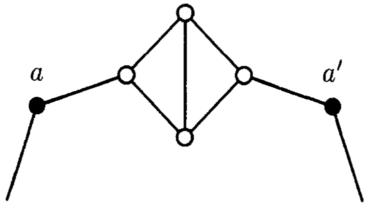

Fig. 2. Consecutive vertices in A_v.

It is immediate that the resulting graph $H(G)$ is 2-connected cubic. To complete the proof, it now suffices to show the following.

Claim. *G is 1-tough if and only if $H(G)$ is 1-tough.*

Proof of the Claim. Suppose first that G is not 1-tough. Then there exists a nonempty set $X \subseteq V(G)$ with $\omega(G - X) > |X|$. Let $Y \subseteq V(H)$ be given by $Y = \bigcup_{v \in X}(A_v \cup B_v)$. It is easily verified that $\omega(H - Y) > |Y|$, and thus H is also not 1-tough.

Conversely, suppose H is not 1-tough. Then there exists a nonempty set $Y \subseteq V(H)$ with $\omega(H - Y) > |Y|$. We will now establish a series of properties (Lemmas 2.1–2.4) which we may assume Y satisfies, since otherwise we may select a nonempty set $Y' \subseteq V(H)$ with $\omega(H - Y') > |Y'|$ satisfying the desired properties.

Lemma 2.1. *We may assume, for each $y \in Y$, that $N_H(y)$ is an independent set.*

Proof. Suppose $N_H(y)$ is not independent for some $y \in Y$. Set $Y' = Y - \{y\}$, so that $|Y'| = |Y| - 1$ and since H is cubic, $\omega(H - Y') \geqslant \omega(H - Y) - 1$. In particular, $\omega(H - Y') > |Y'|$.

If $Y' = \emptyset$, then Y is a singleton and thus H is not 2-connected, a contradiction. Otherwise we have a nonempty $Y' \subseteq V(H)$ such that $\omega(H - Y') > |Y'|$ and we can simply iterate this modification to Y until the desired condition holds. □

Since every vertex in $H_v - (A_v \cup B_v)$ belongs to a triangle, Lemma 2.1 implies that we may assume $Y \subseteq \bigcup_{v \in V(G)}(A_v \cup B_v)$.

Lemma 2.2. *We may assume, for all $v \in V(G)$, that $A_v \cap Y$ (resp., $B_v \cap Y$) is either A_v (resp., B_v) or $\emptyset$.*

Proof. Suppose $A_v \cap Y$ is neither A_v nor $\emptyset$. Then there are two "consecutive" vertices $a, a' \in A_v$ (see Fig. 2) with $a \in Y$, $a' \notin Y$. Set $Y' = Y \cup \{a'\}$. It is immediate that $|Y'| = |Y| + 1$ and $\omega(H - Y') \geqslant \omega(H - Y) + 1$, and thus we have a nonempty $Y' \subseteq V(H)$ such that $\omega(H - Y') > |Y'|$. We simply iterate this modification to Y until $A_v \cap Y = A_v$.

The proof for B_v is identical. □

We will call $v \in V(G)$ a *split vertex* if exactly one of A_v, B_v belongs to Y. In a moment we will show we may assume there are no split vertices. First we need the following result.

Lemma 2.3. *Let $v \in V(G)$. If $A_v \nsubseteq Y$ and $B_v \subseteq Y$, then we may assume, for all $w \in N_G(v)$, that $B_w \subseteq Y$.*

Proof. Suppose $A_v \nsubseteq Y$, $B_v \subseteq Y$, but $B_w \nsubseteq Y$ for some $w \in N_G(v)$. Set $Y' = Y \cup A_v$. We find $|Y'| = |Y| + d_G(v)$ and $\omega(H - Y') \geqslant \omega(H - Y) + d_G(v)$, and thus we have a nonempty $Y' \subseteq V(H)$ such that $\omega(H - Y') > |Y'|$. Now simply iterate the modification of Y until the desired condition holds. □

Lemma 2.4. *We may assume there are no split vertices.*

Proof. Suppose $S = \{v \in V(G) \mid v \text{ is a split vertex}\} \neq \emptyset$. Consider any component C of $\langle S \rangle$ in G, and let $V(C) = \{v_1, v_2, \ldots, v_m\}$. By Lemma 2.3 we may assume, without loss of generality, that $A_{v_i} \nsubseteq Y$, $B_{v_i} \subseteq Y$, for $i = 1, 2, \ldots, m$. Note that if $wv_i \in E(G)$ for some i, $1 \leqslant i \leqslant m$, and $w \notin V(C)$, then by Lemma 2.3, $A_w, B_w \subseteq Y$. Also note that $\bigcup_{i=1}^m (H_{v_i} - B_{v_i})$ induces a subgraph in H with exactly $d_G(v_1) + \cdots + d_G(v_m)$ components. Now set $Y' = Y \backslash (\bigcup_{i=1}^m B_{v_i})$. Suppose $Y' = \emptyset$. If $V(C) \neq V(G)$, then G is disconnected by the above observation on edges wv_i. On the other hand, if $V(C) = V(G)$, then $\omega(H - Y) = \sum_{v \in V(G)} d_G(v) = |Y|$, a contradiction. Hence $Y' \neq \emptyset$. Thus, we have

$$|Y'| = |Y| - \sum_{i=1}^{m} d_G(v_i) \geqslant 1. \tag{1}$$

By Lemma 2.3, we also have

$$\omega(H - Y') = \omega(H - Y) - \left(\sum_{i=1}^{m} d_G(v_i) - 1 \right). \tag{2}$$

Using (1), (2), and $\omega(H - Y) > |Y|$, we find that Y' is a nonempty subset of $V(H)$ such that

$$\omega(H - Y') > |Y| - \sum_{i=1}^{m} d_G(v_i) + 1 = |Y'| + 1 > |Y'|.$$

Now simply iterate the above modification to Y for every component C of $\langle S \rangle$. □

Finally, let $X = \{v \in V(G) \mid A_v, B_v \subseteq Y\}$. Since there are no split vertices, it is easy to check that X is a nonempty subset of $V(G)$ such that $\omega(G - X) > |X|$ and so G is not 1-tough.

This proves the Claim, and completes the proof of Theorem 1.8. □

Proof of Theorem 1.9. We will reduce 1-TOUGH for 2-connected, cubic graphs to t-TOUGH for $3t$-regular graphs, where $t \geqslant 1$ is an integer. Let G be any 1-tough, 2-connected cubic graph. By a well-known theorem of Petersen [21], G can be edge-partitioned into a 1-factor and a 2-factor.

Construct $H = H(G)$ as follows. Each vertex in G is replaced by a K_t in H. Each edge in the 1-factor in G is replaced by a matching (henceforth called an *m-join*) between the corresponding K_t's in H, while each edge in the 2-factor in G is replaced by a complete bipartite join (henceforth called a *c-join*) between the corresponding K_t's. It is immediate that H is $3t$-regular.

We next want to show that H is $2t$-connected. Since G is 2-connected, disconnecting G requires removing at least two vertices, one vertex and a nonincident edge, or two independent edges. Thus, disconnecting H requires removing at least two K_t's, removing one K_t and breaking a nonincident m-join, or breaking two independent m-joins. Note that "breaking an m-join" means removing enough vertices in the m-join to eliminate all edges in the join without completely removing either K_t; obviously this requires removing a total of at least t vertices in the two K_t's. It follows that any cutset in H must contain at least $2t$ vertices, and thus H is $2t$-connected.

To complete the proof, we now show G is 1-tough if and only if H is t-tough. If G is not 1-tough, there exists a cutset $X \subseteq V(G)$ with $\omega(G - X) > |X|$. Let $Y \subseteq V(H)$ consist of the K_t's corresponding to the vertices in X. It is easy to see that Y is a cutset and $\omega(H - Y) = \omega(G - X) > |X| = |Y|/t$, and thus H is not t-tough.

Conversely, suppose H is not t-tough. Then there exists a cutset $Y \subseteq V(H)$ with $\omega(H - Y) > |Y|/t$. We now establish the following.

Claim. *We may assume each K_t in H is entirely contained in Y or entirely disjoint from Y (i.e., no K_t in H is "split" by Y).*

Assuming, we have established the Claim, let $X \subseteq V(G)$ denote the vertices in G corresponding to the K_t's in Y. Then X is a cutset in G and $\omega(G - X) = \omega(H - Y) > |Y|/t = |X|$, and thus G is not 1-tough as desired.

To prove the claim, consider a cutset $Y \subseteq V(H)$ such that $\omega(H - Y) > |Y|/t$ and Y splits as few K_t's in H as possible. If Y splits no K_t's there is nothing to prove, so suppose A is a K_t split by Y, and let B denote the K_t which is m-joined to A. We now consider several cases.

Case 1: B is not split by Y. Let $Y' = Y - (A \cap Y)$. Then $|Y'| < |Y|$ while $\omega(H - Y') = \omega(H - Y)$, since A is still c-joined to the same K_t's as was $A - Y$. Thus, we have $|Y'|/(\omega(H - Y')) < |Y|/(\omega(H - Y)) < t$, or $\omega(H - Y') > |Y'|/t$. Since Y is a cutset and $\omega(H - Y') = \omega(H - Y)$, Y' is also a cutset in H. Since Y' splits fewer K_t's than Y, this violates the optimality of Y.

Case 2.1: B is split by Y and $|A \cap Y| + |B \cap Y| < t$. Set $Y' = Y - (A \cap Y) - (B \cap Y)$. Then $\omega(H - Y') = \omega(H - Y)$, since $A - Y$ and $B - Y$ belong to the same component of $H - Y$, and A (resp, B) is c-joined to its adjacent K_t's besides B (resp, A). Thus, we get $|Y'|/(\omega(H - Y')) < |Y|/(\omega(H - Y)) < t$, or $\omega(H - Y') > |Y'|/t$. Since Y is a cutset and $\omega(H - Y') = \omega(H - Y)$, Y' is also a cutset in H. Again, Y' splits fewer K_t's than Y, and this violates the optimality of Y.

Case 2.2: B is split by Y and $|A \cap Y| + |B \cap Y| \geqslant t$. Let $Y' = Y - (A \cap Y) - Z$, where $Z \subseteq B \cap Y$ is any subset with $|Z| = t - |A \cap Y| > 0$. Since H is $2t$-connected and Y is a cutset in H, we have $|Y'| = |Y| - t \geqslant 2t - t = t$. Note that $\omega(H - Y') \geqslant \omega(H - Y) - 1$, since we might still lose one component by pulling together the two components containing $A - Y$ and $B - Y$, but nothing more. Since $\omega(H - Y) > |Y|/t$, we get $|Y'|/(\omega(H - Y')) \leqslant (|Y| - t)/(\omega(H - Y) - 1) < t$, or $\omega(H - Y') > |Y'|/t \geqslant 1$. Thus, Y' is a cutset in H. Since Y' splits fewer K_t's than Y, this violates the optimality of Y.

This proves the Claim, and thereby proves Theorem 1.9. □

We conclude this section by remarking that a possible approach toward proving Conjecture 1.12 is to interchange the roles of the m-joins and c-joins in the construction of $H(G)$ in the proof of Theorem 1.9. However at present, we have not been able to verify the Claim in the proof of Theorem 1.9 for $H(G)$ constructed in this new way.

3. Concluding remarks

There remain a number of interesting questions concerning the complexity of recognizing special classes of tough graphs. Recall the following well-known conjecture.

Barnette's Conjecture. Every 3-*connected, cubic, planar, bipartite graph is hamiltonian.*

It is well known that if any of the hypotheses in this conjecture are dropped, the conclusion that the graph is hamiltonian need not follow. Thus, it seems interesting to consider the complexity of recognizing 1-tough graphs when one or more of the hypotheses in Barnette's Conjecture are dropped.

It is easy to see that every 3-connected cubic graph is 1-tough. On the other hand, there are 2-connected, cubic, planar, bipartite graphs which are not 1-tough (see, e.g., [2]). The complexity of recognizing 1-tough graphs remains open for the following classes of graphs:

- 2-connected, cubic, planar, bipartite graphs,
- 2-connected, cubic, planar graphs,
- 2-connected, cubic, bipartite graphs,
- 2-connected, planar, bipartite graphs,
- 2-connected, planar graphs.

Tutte [25] has shown that every 4-connected planar graph is hamiltonian and Thomassen [24] has shown that every such graph is hamiltonian connected. On the other hand, there exist 3-connected, planar, bipartite graphs which are not 1-tough (e.g., the Herschel graph [5, p. 53]). The complexity of recognizing 1-tough graphs remains open for the following classes.

- 3-connected, planar, bipartite graphs,
- 3-connected, planar graphs,
- 3-connected, bipartite graphs.

It is interesting to note that the complexity of recognizing hamiltonian graphs is known to be NP-hard for all of the above classes except possibly 3-connected, planar, bipartite graphs [1, 16].

Finally, let us focus on the class of planar graphs. As indicated above, we do not know the complexity of recognizing 1-tough, planar graphs. However, the next result might yield a clue. It follows from theorems in [13, 22].

Theorem 3.1 (Dillencourt, Schmeichel and Bloom [13, 22]). *Let G be a planar graph on at least 5 vertices. Then G is 4-connected if and only if $\omega(G-X) \leqslant |X|-2$, for all cutsets $X \subseteq V(G)$ with $|X| \geqslant 3$.*

Since 4-connected graphs can be recognized in polynomial time, it follows that for planar graphs G, it can be determined in polynomial time whether $\omega(G-X) \leqslant |X|-2$, for all cutsets $X \subseteq V(G)$ with $|X| \geqslant 3$. To determine if G is 1-tough, one needs to decide the superficially similar inequality $\omega(G-X) \leqslant |X|$, for all cutsets $X \subseteq V(G)$. Perhaps this suggests that recognizing 1-tough, planar graphs can be done in polynomial time.

Dillencourt [12] has also inquired about the complexity of recognizing 1-tough, maximal planar graphs, noting that recognizing hamiltonian, maximal planar graphs is NP-hard. All we know is that there exist maximal planar graphs which are not 1-tough.

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